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Simple Formulation to Predict Thermal Postbuckling Load of Circular Plates

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Nomenclature

a	=	radius of the circular plate
c	=	coefficient in Eq. (15)
D	=	plate flexural rigidity $[=Et^3/12(1-\nu^2)]$
E	=	Young's modulus
N_r	=	uniform radial edge compressive load per unit length
$N_{r_{cr}}$	=	linear buckling load
$N_{r_{NL}}$	=	total uniform radial edge compressive load per unit length
N_{r_T}	=	uniform radial edge tensile load per unit length developed due to large lateral displacements
r	=	radial coordinate
t	=	thickness of the circular plate
u	=	radial displacement
w	=	lateral displacement
w_0	=	central (maximum) lateral displacement of the circular plate
α	=	coefficient of linear thermal expansion
ΔT	=	temperature rise from the stress free temperature
ε_r, ε_θ	=	in-plane strains
λ	=	ratio of the thermal postbuckling to linear buckling load
ν	=	Poisson ratio

Introduction

THIN circular plates are commonly used structural members in large aerospace structures. During their service condition, these plates are subjected to thermal loads, arising from the aerodynamic and/or solar heating. Prediction of the thermal postbuckling load of such plates serves the purpose of arriving at the cost effective designs.

Analytical treatment of the postbuckling of uniform, isotropic columns, circular and rectangular plates subjected to mechanical loads can be seen in the books of Thompson and Hunt [1] and Dym [2]. The versatile finite element and the classical Rayleigh–Ritz analyses are presented for the thermal postbuckling analysis of uniform columns [3] and tapered columns [4] with immovable ends in the axial direction and uniform circular plates [5] with immovable edges in the radial direction. In all these studies, estimation of the postbuckling load involves solving either the corresponding nonlinear differential equations or obtaining approximate solutions from the nonlinear energy formulations and is generally complex. As such, a simple formulation to predict the thermal postbuckling load of uniform columns is presented by Rao and Raju [6]. Applicability of this simple formulation to square plates has been demonstrated to predict the thermal postbuckling load with the immovable in-plane displacements normal to the edges of the plate [7].

The aim of the present Note is to present a similar formulation to obtain the thermal postbuckling behavior of uniform thin circular plates, with radially immovable edges. The advantage of the present simple formulation is that it requires only the knowledge of the uniform radial edge tensile load developed due to the large axisymmetric lateral displacements and the corresponding linear buckling load. It is to be noted here that the evaluation of the uniform radial edge tensile load for the circular plates is more involved as both the radial and circumferential in-plane strains are dependent on the radial displacement even for the axisymmetric case, and the corresponding nonlinear differential equation is relatively difficult to solve for the same, when compared to the evaluation of the axial end tensile load for the columns and biaxial state of in-plane uniform edge tensile loads for the square plates [7].

In the following sections the simple formulation proposed to obtain the thermal postbuckling load of circular plates and the method for evaluating the radial edge tensile load developed due to the large lateral displacements are presented. The values of the corresponding linear buckling loads are taken from the literature for both the simply supported and clamped uniform circular plates.

Present Formulation

If the circular plate of radius a, with edges immovable in the radial direction, is heated to a temperature ΔT from the stress free state, an equivalent uniform compressive radial edge load N_r is developed in the plate. When the temperature becomes the critical temperature ΔT_{cr}, the plate just buckles because of the critical compressive uniform radial edge load $N_{r_{cr}}$ developed. If the temperature ΔT is further raised, lateral displacements of the plate take place and an additional uniform tensile radial edge load N_{r_T} is developed because of the large lateral displacements. This N_{r_T}, for a particular central (maximum) lateral displacement, allows the plate to take more thermal load beyond the critical load or in other words the plate can withstand more equivalent compressive uniform radial edge load N_r beyond $N_{r_{cr}}$. Thus the total equivalent compressive uniform radial edge load carrying capacity of the circular plate $N_{r_{NL}}$, which is the postbuckling load, can be mathematically represented, in the nondimensional form, as

$$\bar{N}_{r_{NL}} = \bar{N}_{r_{cr}} + \bar{N}_{r_T} \tag{1}$$

where each term in Eq. (1) is nondimensionalized as

$$\bar{N}_r = \frac{N_r a^2}{D} \tag{2}$$

The thermal load equivalent of $\bar{N}_r$ is [5]

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$$\bar{N}_r = 12(1+\nu)\alpha\Delta T a^2/t^2 \tag{3}$$

Uniform Radial Edge Tension Developed due to Large Axisymmetric Lateral Displacements

The uniform radial edge tensile load developed in the circular plate with radially immovable edges is calculated using the following procedure. First, the immovability condition at the edge of the circular plate is relaxed and the plate is given a large axisymmetric lateral displacement. Because of this, the edge of the plate will have a uniform inward radial edge displacement. To compensate this inward radial edge displacement, a uniform radial tensile load is applied on the edge of the plate. This gives an outward radial edge displacement. The magnitude of the uniform radial edge tensile load in the plate, to satisfy the radially immovable edge condition, is obtained using the condition that the magnitudes of the inward radial edge displacement due to large axisymmetric lateral displacement and the outward radial edge displacement due to the applied uniform radial edge tensile load are equal. The expressions for the inward and outward radial edge displacements are derived in this section.

The nonlinear strain-displacement relations of a circular plate undergoing large lateral axisymmetric displacements are

$$\varepsilon_r = \frac{\mathrm{d}u}{\mathrm{d}r} + \frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2 \tag{4}$$

and

$$\varepsilon_\theta = \frac{u}{r} \tag{5}$$

It can be assumed here without loss of rigor, that the lateral axisymmetric displacement of the plate represents a very shallow shell of revolution with a near zero Gaussian curvature and hence its surface is developable. It may also be noted here that the deformed configuration of the simply supported circular plate represents a very shallow spherical shell. A brief explanation on the effect of this assumption is given in the next section.

Because of the developability of the laterally deformed surface of the circular plate, the expressions and magnitude of both N_r and N_θ are given by

$$N_r = \frac{Et}{1-\nu^2}\left[\frac{\mathrm{d}u}{\mathrm{d}r} + \frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2 + \nu\frac{u}{r}\right] = 0 \tag{6}$$

and

$$N_\theta = \frac{Et}{1-\nu^2}\left\{\frac{u}{r} + \nu\left[\frac{\mathrm{d}u}{\mathrm{d}r} + \frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2\right]\right\} = 0 \tag{7}$$

Equations (6) and (7) give the relation for the inward radial edge displacement u_I as

$$u_I = -\frac{1}{2}\int_0^a \left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2 \mathrm{d}r \tag{8}$$

The outward radial edge displacement u_T for an applied uniform radial edge tensile load can be calculated by considering a sector of the plate with a subtended angle θ at the center. In consideration of the equilibrium of the sector, u_T is obtained as

$$u_T = \frac{N_{rT}a}{Et} \tag{9}$$

The uniform radial edge tensile load should be such that the magnitudes of u_I and u_T are equal and gives its value as

$$N_{r_T} = \frac{Et}{2a}\int_0^a \left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2 \mathrm{d}r \tag{10}$$

or, in the nondimensional form

$$\bar{N}_{r_T} = \frac{6a(1-\nu^2)}{t^2}\int_0^a \left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2 \mathrm{d}r \tag{11}$$

Numerical Results and Discussion

From Eq. (11) the uniform radial edge tensile load $\bar{N}_{r_T}$ developed in the circular plate with radially immovable edges can be obtained by assuming suitable admissible functions for the lateral displacement w, which must satisfy the geometric boundary conditions. The value of the Poisson ratio ν is taken as 0.3 for obtaining the numerical results. Both the simply supported and clamped uniform circular plates are considered in the present note.

The following admissible functions F_1, F_2, and F_3 for the lateral displacement w are considered in the present study:

$$F_1 = w_0\left[1 - \left(\frac{r}{a}\right)^2\right]^n \tag{12}$$

$$F_2 = w_0 \cos^n \frac{\pi r}{2a} \tag{13}$$

and

$$F_3 = w_0\left[1 + C_1\left(\frac{r}{a}\right)^2 + C_2\left(\frac{r}{a}\right)^4\right] \tag{14}$$

The first two functions F_1 and F_2 satisfy the geometric boundary conditions, whereas the function F_3, taken from Yamaki [8], satisfies both the geometric and natural boundary conditions. For the functions F_1 and F_2 the values of $n = 1$ and 2 represent the simply supported and clamped boundary conditions of the circular plate, respectively, and for the function F_3 the values of $C_1 = -(6+2\upsilon)/(5+\upsilon)$, $C_2 = (1+\upsilon)/(5+\upsilon)$ represent the simply supported boundary conditions and $C_1 = -2$, $C_2 = 1$ represent the clamped boundary conditions of the circular plate, respectively.

Accurate values of linear buckling load parameters $\bar{N}_{r_{\mathrm{cr}}}$ are readily available in the literature [9] and the numerical values are 4.1978 for the simply supported circular plate and 14.6896 for the clamped circular plate. However, the accuracy of the value of $\bar{N}_{rT}$ obtained from Eq. (11) depends on the assumption of the near zero Gaussian curvature and the accuracy of the assumed admissible function for the lateral displacement. For the present study, the Gaussian curvature evaluated from the curvature-displacement relations is of the order of $10^{-4}/a^2$ in magnitude which is treated as near zero, for thin circular plates ($t/a = 0.01$), whether simply supported or clamped, with the value of $w_0/t = 1.0$. Many of the researchers dealing with the postbuckling study of the plates limited their study up to $w_0/t = 1.0$, and the present work is valid up to the value $w_0/t = 1.0$ for which $t/a = 0.01$ and gives the magnitude of Gaussian curvature $10^{-4}/a^2$ which is treated as near zero.

The ratio of the postbuckling radial edge load to the linear buckling load parameters, which represent the postbuckling load, can be obtained from Eq. (1) using the values of the radial edge tensile load parameters calculated from the admissible functions chosen for the lateral displacement w and the linear buckling load parameters given earlier as

$$\lambda = \frac{\bar{N}_{r\mathrm{NL}}}{\bar{N}_{r_{\mathrm{cr}}}} = 1 + c\left(\frac{b}{t}\right)^2 \tag{15}$$

The values of c for the admissible functions F_1, F_2, and F_3 for the simply supported plate are 1.7430, 1.6044, and 1.5967 and for the clamped circular plate are 0.4533, 0.4587, and 0.4533, respectively. The variation of λ for the simply supported and the clamped plates with w_0/t is given in Table 1 for the three admissible functions chosen along with those given by Raju and Rao [5]. It can be seen from this table that the three admissible functions give more or less the same values of c for a specified boundary condition. However, the results obtained by using the admissible function F_1 for the

Table 1 Variation of λ for uniform circular plates with w_0/t

	Simply supported				Clamped			
	Present study				Present study			
w_0/t	F_1	F_2	F_3	FEM [5]	F_1	F_2	F_3	FEM [5]
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	1.0173	1.0160	1.0160	1.0182	1.0045	1.0046	1.0045	1.0052
0.2	1.0694	1.0642	1.0639	1.0730	1.0181	1.0184	1.0181	1.0210
0.3	1.1561	1.1444	1.1437	1.1645	1.0408	1.0413	1.0408	1.0472
0.4	1.2775	1.2567	1.2555	1.2931	1.0725	1.0734	1.0725	1.0840
0.5	1.4336	1.4011	1.3992	1.4593	1.1133	1.1147	1.1133	1.1314
0.6	1.6244	1.5776	1.5748	1.6637	1.1632	1.1652	1.1632	1.1893
0.7	1.8498	1.7862	1.7824	1.9072	1.2221	1.2248	1.2221	1.2580
0.8	2.1100	2.0269	2.0219	2.1906	1.2901	1.2936	1.2901	1.3373
0.9	2.4048	2.2996	2.2934	2.5151	1.3672	1.3716	1.3672	1.4275
1.0	2.7343	2.6045	2.5968	2.8818	1.4533	1.4588	1.4533	1.5286

simply supported circular plate are nearer to those obtained using the finite element method [5] (FEM) as this function alone satisfies the condition, for the deformed configuration of the simply supported circular plate to be spherical, namely, the meridional and circumferential curvatures are the same. For the clamped circular plates the solutions based on the chosen admissible functions give the same accuracy.

Conclusions

A simple formulation is presented in this Note to predict the thermal postbuckling load of uniform, isotropic circular plates with radially immovable edges, having simply supported or clamped boundary conditions. There is a satisfactory agreement of the present numerical results obtained from the present formulation with the converged finite element solutions, demonstrating its effectiveness. Simplicity, which is the main emphasis in the present Note, is achieved by introducing a simplifying but justifiable assumption in evaluating the uniform radial edge tensile load. The numerical results presented indicate that this assumption not only simplifies the present formulation but also gives numerical results within the engineering accuracy.

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